$$\int_{t=a}^{b} \bar{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

for space curve $\bar{r}(t) = \langle x(t), y(t), z(t) \rangle$

Arc length

- The arc length of a curve should be computable by

1. approximate curve by straight line segments
2. length of each segment add to approximate length of curve.

→ limit these, using more and more line segments

9/17/2021

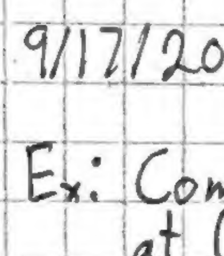
Ex: Compute the tangent line for $\bar{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos($ at $(\sqrt{3}, 1, 2)$.

Sol: $\bar{r}'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(2t) \rangle$

Need t for the point $(\sqrt{3}, 1, 2)$.

$$\begin{cases} 2\cos(t) = \sqrt{3} \\ 2\sin(t) = 1 \\ 4\cos(2t) = 2 \end{cases} \Rightarrow \begin{cases} \cos(t) = \sqrt{3}/2 \\ \sin(t) = 1/2 \\ \cos(2t) = 1/2 \end{cases} \Rightarrow t = \frac{\pi}{6} + 2k\pi$$

Check: $\cos\left(2\cdot\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.



$\therefore$ The tangent vector at the point given is

$$\bar{r}'(\pi/6) = \langle -2\sin(\pi/6), 2\cos(\pi/6), -8\sin(2\cdot\pi/6) \rangle$$
$$= \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

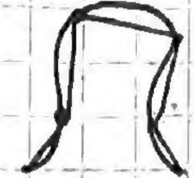
$\therefore$ The tangent has vector equation.

$$\bar{u}(t) = \bar{p} + t\bar{r}'(\pi/6) = \langle \sqrt{3}, 1, 2 \rangle + t\langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$
$$= \langle \sqrt{3} - t, 1 + \sqrt{3}t, 2 - 4\sqrt{3}t \rangle$$

Section: 13. Arc length.

Recall: the arc length of space curve $\bar{r}(t)$ between times $t=a$ and b is

$$S = \int_{t=a}^{b} |\bar{r}\,'(t)|\, dt$$



$\bar{r}(t)$ is a space curve in $\mathbb{R}^3$: $\bar{r}(t) = \langle x(t), y(t) \rangle$

$$\bar{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

$$|\bar{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

for a plane curve (like in Calc II)

arc length $\nearrow$ $S = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex. Find the arc length of $\bar{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$ in $0 \leq t \leq \frac{\pi}{4}$

Sol. Arc lenght has formula ; $S = \int_{t=a}^{b} |\bar{r}\,'(t)|\, dt$

now $a=0$ and $b = \pi/4$

$$\bar{r}\,'(t) = \left\langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \right\rangle$$

$$= \langle -\sin(t), \cos(t), -\tan(t) \rangle$$

$$|\bar{r}\,'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (-\tan(t))^2}$$

$$= \sqrt{\sin^2(t) + \cos^2(t) + \tan^2(t)}$$

$$= \sqrt{1 + \tan^2(t)}$$

$$= \sqrt{\sec^2(t)} = |\sec(t)| = \sec(t) \text{ on } 0 \leq t \leq \frac{\pi}{4}$$

$\sqrt{x^2} = |x|$

$$S = \int_{t=a}^{b} |\bar{r}'(t)|\,dt$$

$$= \int_{t=0}^{\pi/4} \sec(t)\,dt$$

$$\Big[\ln|\sec(t) + \tan(t)|\Big]_{t=0}^{\pi/4}$$

$$\ln\left|\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|\sqrt{2}+1| - \ln|1+0|$$

$$\boxed{= \ln(1+\sqrt{2})}$$

Ex. Compute the arc length of $\bar{r}(t) = \langle \cos(t), \sin(t), t^2 \rangle$ on $\frac{\pi}{3} \le t \le \pi$

Sol.: $\bar{r}'(t) \langle -\sin(t), \cos(t), 2t \rangle$

$$\therefore |\bar{r}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (2t)^2}$$

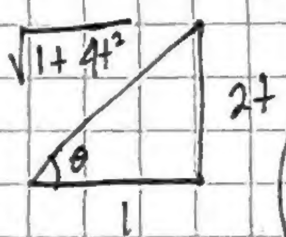
$$= \sqrt{\sin^2(t) + \cos^2(t) + 4t^2}$$

$$= \sqrt{1+4t^2}$$

$a = \frac{\pi}{3}$ $\quad \therefore\ s = \int_{t=a}^{b} |\bar{r}'(t)|\,dt$

$b = \pi$

$$S = \int_{t=\frac{\pi}{3}}^{\pi} \sqrt{1+4t^2}\,dt = \int_{t=\frac{\pi}{3}}^{\pi} \frac{1}{2}\sqrt{1+4t^2}\,2dt$$



$\tan(\theta) = \frac{2t}{1}$

$\sec(\theta) = \frac{\sqrt{1+4t^2}}{1}$

$\sec^2(\theta)\,d\theta = 2dt$

$$\int_{t=\frac{\pi}{3}}^{\pi} \frac{1}{2}\sec(\theta)\sec^2(\theta)\,d\theta = \frac{1}{2}\int_{t=\frac{\pi}{3}}^{\pi} \sec(\theta)(1+\tan^2(\theta))\,d$$

$$= \frac{1}{2}\left(\int_{t=\frac{\pi}{3}}^{\pi} \sec(\theta)\,d\theta + \int_{t=\frac{\pi}{3}}^{\pi} \sec(\theta)\tan^2(\theta)\,d\theta\right)$$

to compute $\int \sec(\theta)\tan^2(\theta)\,d\theta$

$u = \tan(\theta)$ $\quad dv = \sec(\theta)\tan(\theta)\,d\theta$

$du = \sec^2(\theta)\,d\theta$ $\quad v = \sec(\theta)$

$$\int u\,dv = uv - \int v\,du$$

$$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\sec^2(\theta)\,d\theta$$

$$\therefore \int \sec^3(\theta)\,d\theta$$

$$= \int \sec(\theta)\,d\theta + \sec(\theta)\tan(\theta)$$

$$- \int \sec^3(\theta)\,d\theta$$

$\therefore 2\int \sec^3(\theta)\,d\theta = \ln|\sec(\theta)+\tan(\theta)| + \sec(\theta)\tan(\theta)$

Finally: $s = \frac{1}{2}\int_{t=\frac{\pi}{3}}^{\pi} \sec^3(\theta)\,d\theta$

$$= \frac{1}{4}\Big[\ln|\sec(\theta)+\tan(\theta)| + \sec(\theta)\tan(\theta)\Big]_{t=\frac{\pi}{3}}^{\pi}$$

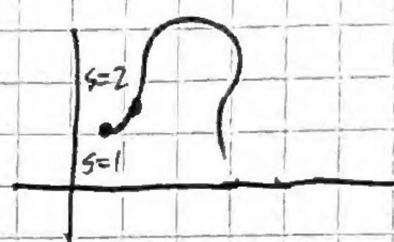
$$= \frac{1}{4}\Big[\ln\left|\sqrt{1+4t^2}+2t\right| + \sqrt{1+4t^2}\cdot 2t\Big]_{t=\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{4}\Big(\ln\left|\sqrt{1+4\pi^2}+2\pi\right| + 2\pi\sqrt{1+4\pi^2}$$

$$- \ln\left|\sqrt{1+\frac{4}{9}\pi^2}+\frac{2}{3}\pi\right| + \frac{2}{3}\pi\sqrt{1+\frac{4}{9}\pi^2}\Big)$$

The arc length is the "most natural parameter" for a curve.

In particular if we make a parameterization of the curve with arc length s at time s (measured from the point), then that parameterization has unit speed.



Example next time...